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ABSTRACT

The critical behavior of the continuous-spin Ising model is studied by high temperature methods and compared with renormalization group results. The critical exponent inequality $\delta \geq \Delta/(\Delta-\gamma)$ is proven and used to show that $2\Delta < d\nu + \gamma$ requires $\gamma(\delta+1)/(\delta-1) < d\nu$.

AN ANALYSIS OF THE CONTINUOUS-SPIN, ISING MODEL*

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Since the time when the study of relations between the various critical indices was systemitized,[1] these indices have been classed into groups. First, I remind you of some notation. If χ is the magnetic susceptibility, M the magnetization, C_H the specific heat at constant magnetic field, and ξ the correlation length, then near the critical point, temperature, $T = T_c$, and magnetic field, $H = 0$, for an Ising model on a d-dimensional, rigid, regular space-lattice we expect, $T > T_c$, $H = 0$,

$$\chi \sim A_+(T-T_c)^{-\nu}, \quad \xi \sim D_+(T-T_c)^{-\nu},$$

$$-\frac{\partial^2\chi}{\partial H^2} \approx B_+(T-T_c)^{-\gamma-2\Delta}, \quad C_H \propto (T-T_c)^{-\alpha}, \tag{1}$$

$T = T_c$,

$$M \propto H^{1/\delta}, \quad \langle\sigma_0\sigma_{\vec r}\rangle\Big|_{H=0} \propto r^{-d+2-\eta} \tag{2}$$

where $\langle\sigma_0\sigma_{\vec r}\rangle\Big|_{H=0}$ is the spin-spin correlation function between a spin σ at the origin and one at $\vec r$ in zero magnetic field.

$T < T_c$, $H = 0$,

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$$\chi \simeq A_-(T_c-T)^{-\gamma'}, \quad \xi \simeq D_-(T_c-T)^{-\nu'}$$

$$-\frac{\partial^2\chi}{\partial H^2} \simeq B_-(T_c-T)^{-\gamma'-2\Delta'}, \quad C_H \propto (T_c-T)^{-\alpha'}$$

$$M \propto (T_c-T)^{\beta}. \tag{3}$$

In terms of this notation, a selection of the relations between the critical indices (α, γ, δ, etc.) would be:

single temperature region,

$$\alpha' + 2 \quad + \gamma' = 2; \tag{4}$$

critical isotherm plus a single temperature region,

$$\alpha' + \beta(1+\delta) = 2,$$

$$\delta = \Delta/(\Delta-\gamma); \tag{5}$$

two temperature regions

$$\gamma = \gamma', \quad \alpha = \alpha',$$

$$\Delta = \Delta'; \tag{6}$$

relations involving correlation exponents,

$$\gamma = (2-\eta)\nu,$$

$$\gamma' = (2-\eta)\nu'; \tag{7}$$

and relations involving the spatial dimension or hyperscaling,

$$d\nu = 2 - \alpha,$$

$$2 - \eta = d(\delta-1)/(\delta+1),$$

$$2\Delta = d\nu + \gamma. \tag{8}$$

On the numerical evidence, the hyperscaling relations (8) were the least well supported and those of (6) suffered initially from the weakness of the accuracy in the $T < T_c$ numerical results. Many of these relations have been proven to be rigorous inequalities, e.g.,[2-6]

$$\alpha' + 2\beta + \gamma' \geq 2, \quad \gamma \geq (2-\eta)\nu,$$

$$d\nu + \gamma \geq 2\Delta, \quad \delta \geq \Delta/(\Delta-\gamma),$$

$$\alpha' + \beta(1+\delta) \geq 2. \tag{9}$$

In order to understand what was going on, and to gain a deeper understanding of these exponent relations, general ideas that related them to scaling properties were put forth.[7,8] These ideas were further developed and extended by the use of field theoretic methods[9,10] to yield the renormalization group theory of critical phenomena, which rests on the renormalization group hypothesis[11,12] and has all the index relations (4-8) as a consequence for $d \leq 4$.

Now the trouble starts when one compares the results of the renormalization group theory of critical phenomena with those of the high-temperature series numerical computations. These high temperature series results yield, for example[13-15]

$$\gamma = 1.250 \pm 0.003,$$

$$\nu = 0.638 \begin{smallmatrix} + 0.002 \\ - 0.001 \end{smallmatrix},$$

$$\Delta = 1.563 \pm 0.003, \tag{10}$$

and[16] for the renormalization group equality (8),

$$2\Delta - d\nu - \gamma = - 0.028 \pm 0.003, \ d = 3,$$

$$= - 0.302 \pm 0.038, \ d = 4. \tag{11}$$

These results show small but persistent deviations[17] from the expected renormalization group results,[14,18] in three dimensions,

$$\gamma = 1.241 \pm 0.004, \quad \nu = 0.630 \pm 0.002,$$

$$2\Delta - d\nu - \gamma = 0. \tag{12}$$

Other analyses of the high-temperature series coefficients are to be found in this volume.

To study this discrepancy in detail, I prefer to put it in a broader context[11,12,19] and consider the continuous-spin Ising model which has both the spin-½, Ising model and Euclidean, Boson, quantum field theory as limiting cases.

The partition function for this model is

$$Z(H) = M^{-1} \int_{-\infty}^{+\infty}\cdots\int \prod_{\vec{i}=1}^{N} d\phi_{\vec{i}} \exp\left[-\tfrac{1}{2}v \sum_{\vec{i}=1}^{N} \left\{ \frac{2d}{q} \sum_{\{\vec{\delta}\}} \frac{(\phi_{\vec{i}}-\phi_{\vec{i}+\vec{\delta}})^2}{a^2} + m_o^{\,2}:\phi_{\vec{i}}^{\,2}: + \frac{2}{4!}g_o:\phi_{\vec{i}}^{\,4}: \right\} + \sum_{\vec{i}} H_{\vec{i}}\phi_{\vec{i}} \right] , \tag{13}$$

where a is the lattice spacing, $v \propto a^d$ is the specific volume per lattice site, q is the lattice coordination number, $\{\vec{\delta}\}$ is one-half the set of nearest neighbor sites, and $H_{\vec{j}}$ is the magnetic field at site $\vec{i}$. This model looks like a lattice-cutoff model field theory. If we perform the usual amplitude (Z_3) and mass renormalizations ($m_o^{\,2} = m^2 + \delta m^2$), then we can rewrite (13) as

$$Z(\tilde{H}) = \tilde{M}^{-1} \int_{-\infty}^{+\infty}\cdots\int \prod_{\vec{i}=1}^{N} d\sigma_{\vec{i}} \exp\left[\sum_{\vec{i}} \left\{ K \sum_{\{\vec{\delta}\}} \sigma_{\vec{i}}\sigma_{\vec{i}+\vec{\delta}} - \tilde{g}_o\sigma_{\vec{i}}^{\,4} - \tilde{A}\sigma_{\vec{i}}^{\,2} + \tilde{H}\sigma_{\vec{i}} \right\}\right] . \tag{14}$$

where the relation between the field theory language of (13) and the statistical mechanical language of (14) is

$$\tilde{g}_o = g_o K^2 q^2 a^4/(96 d^2 v) \propto g_o a^{4-d},$$

$$\tilde{A} = qK(2d + m^2a^2 + \delta m^2 a^2 - \tfrac{1}{2}Ca^2 g_o)/4d,$$

$$\tilde{H} = H_I \left[qKa^2/(2dZ_3 v)\right]^{1/2}. \tag{15}$$

Note that we have added a free parameter, K, and imposed a normalization condition,

$$\langle\sigma^2\rangle_{H=K=0} = 1 = \frac{\int_{-\infty}^{+\infty} dx\, x^2 \exp(-\tilde{g}_o x^4 - \tilde{A}x^2)}{\int_{-\infty}^{+\infty} dx \exp(-\tilde{g}_o x^4 - \tilde{A}x^2)} , \tag{16}$$

which fixes $\tilde{A}$ as a function of $\tilde{g}_0$. Further note that C is the usual $[\phi^-,\phi^+]$ commutator which diverges as a goes to zero for $d \geq 2$. As usual, the renormalization conditions imposed on the two-point function,

$$\Gamma_R^{(2)}(p,-p) = \left\{ v \sum_{\vec{j}=0}^{N-1} \frac{\partial^2 \ln Z(H)}{\partial H_0 \partial H_j}\bigg|_{H=0} \exp\left[-2\pi i \vec{p}\cdot\vec{j} a\right] \right\}^{-1} \tag{17}$$

determine the renormalization constants Z_3 and δm^2. These renormalization conditions are

$$\begin{aligned} \Gamma_R^{(2)}(p,-p) &\sim m^2 + 4\pi^2 p^2 + \cdots, \text{ as } p \to 0, \\ &= \frac{2dZ_3}{qKa^2}\, \chi^{-1}(1 + (2\pi)^2 \xi^2 a^2 p^2 + \cdots, \end{aligned} \tag{18}$$

in terms of

$$\chi = \sum_{\vec{j}=0}^{N-1} [\langle \sigma_0 \sigma_{\vec{j}} \rangle - \langle \sigma_0 \rangle^2] ,$$

$$\xi^2 = \frac{\sum_{\vec{j}=0}^{N-1} j^2 (\langle \sigma_0 \sigma_{\vec{j}} \rangle - \langle \sigma_0 \rangle^2)}{2d\chi} , \tag{19}$$

where the expectation values are determined by the partition function (14). These conditions lead to the relations,

$$\begin{aligned} & m^2 \xi^2 a^2 = 1, \\ & Z_3 = (\chi/\xi^2)(qK/2d). \end{aligned} \tag{20}$$

The object to be studied is the dimensionless, renormalized, coupling constant

$$g = g_R m^{d-4} = \frac{-v}{a^d} \frac{\frac{\partial^4 K}{\partial H^4}}{\chi^2 \xi^d} \sim (1 - T_c/T)^{\gamma + d\nu - 2\Delta} \tag{21}$$

This quantity is bounded as $T \to T_c$ by Schrader's[4] inequality. If it goes to zero, then hyperscaling fails (8) and the corresponding field theory is trivial.[20] If it is finite, then hyperscaling holds.

The conventional wisdom for the behavior of $g(g_0,a)$ is that there is a limiting curve which is smoothly approached as $a \to 0$. By eq. (20) for a fixed, renormalized mass, this limit is equivalent to $\xi \to \infty$ with fixed lattice spacing, i.e., the temperature approaches the critical temperature. This limiting curve is conventionally thought to rise monotonically from zero for $g_0 = 0$ to a finite limit g^* for $g_0 = \infty$. Specifically, the renormalization group hypothesis[11,12,19] is that there exists a unique, non-zero limit as $g_0 \to \infty$ and $a \to 0$ independent of the manner of approach. From this hypothesis, as a statistical-mechanical problem corresponds to $\tilde{g}_0$ fixed, and by eq. (15) $\tilde{g}_0 \propto g_0 a^{4-d}$, we must have $g_0 \to \infty$ as $a \to 0$ for $d < 4$ and so $g \to g^*$. As everything is thought to depend on g, we must, based on this hypothesis, get the same result, i.e., universality, for any $\tilde{g}_0$-fixed, statistical-mechanical model. The hypothesized smoothness and differentiability of the approach to the limit yields the critical index relations.

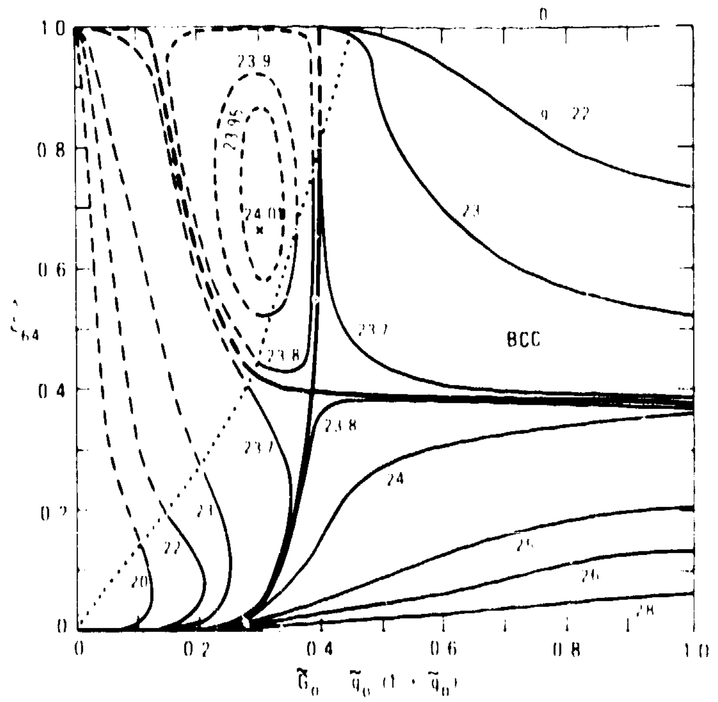

Fig. 1. Contours of the renormalized coupling constant, g, in the $\hat{\xi}_{64}$, G_0 plane for the body-centered-cubic lattice. Here $\hat{\xi}_{64} = \xi^2/(64+\xi^2)$ and $G_0 = g_0/(1+g_0)$. The boldface curve represents $g^* = 23.78$.

Baker and Kincaid[11,19] have made a detailed investigation using high-temperature series methods and concluded on numerical evidence that the renormalization group hypothesis holds for $d = 1, 2$ (known previously[21]) but fails in $d = 3$ and 4 dimensions. The results in three dimensions are particularly interesting as Fig. 1 illustrates. A much richer structure in the g-contour map is found than had been anticipated. The top edge of the figure is a spread-out version of the $g_o = \infty$, $a = 0$ point. They[19] found that $g = g^* = 23.78$ alone did not appear to represent this point and that the g^* contour also extended into the interior and possessed a saddle point. We remark in passing that such a saddle point reconciles these numerical results with Schrader's[22] rigorous results.

Where can we look, theoretically, for the breakdown of hyperscaling in three and four dimensions? Looking back at eq. (8) we note that the occurence of the spatial dimension d appears in association with the relation of a single index such as ν or η for a microscopic property to a thermodynamic index such as α, γ, δ, etc. It is therefore interesting to introduce a thermodynamic coupling constant which replaces the dependence on ξ^d in (21) by a combination of thermodynamic variables. The most obvious move is to use the terms from Sokal's[23] proof of the Josephson inequality to make the replacement

$$\xi^d \to \left(\frac{\partial\xi}{\partial K}\right)^2 \Big/ (C_H\,\xi^2), \tag{22}$$

however, as $C_H \propto \ln(K_c-K)$ for $d = 2$, this replacement would lead to an infinite thermodynamic coupling constant in two dimensions. I prefer to make the replacement

$$\xi^d \to (\chi)^{(\delta+1)/(\delta-1)} \tag{23}$$

One finds directly by use of Fisher's[3] results

$$\gamma \le (2-\eta)\nu, \quad (2-\eta) \le d(\delta-1)/(\delta+1) \tag{24}$$

that

$$d\nu \ge \gamma(\delta+1)/(\delta-1) \tag{25}$$

Hence if we select

$$g_T = \frac{\dfrac{\partial^2\chi}{\partial H^2}}{\chi^2\,\chi^{(\delta+1)/(\delta-1)}} \tag{26}$$

Then, including a dimension and lattice dependent constant, [illegible], related[3] to the amplitude of the decay of the two-spin, correlation function with distance for $T = T_c$ in zero magnetic field, we may conclude

$$\Omega g_T \geq g, \quad (T \to T_c). \tag{27}$$

Since g is bounded from above[4] and goes to zero if hyperscaling fails, and, as we shall see below, since $A_+ \neq 0$ and $B_+ \neq 0$, g_T is not zero, although it could become infinite, we conclude that it is sufficient for (25) to be a strict inequality for

$$2\Delta < d\nu + \gamma. \tag{28}$$

That is to say, if one of the hyperscaling relations (8) fails [here (28)] then necessarily the others [here we will only see (25)] fail as well. Certainly this result is expected,[24] if the non-hyperscaling relations continue to hold. We remark that numerically g_T is finite for the cases tested (e.g., $d = 2,3,\infty$) within error.

Now, to show that (26) does not go to zero as $T \to T_c$, consider

$$F(\tau) = \frac{M-\tau}{(1-\tau^2)} = (\chi-1)\,\tau + \left(\frac{\partial^2\chi}{\partial H^2} + \frac{4}{3}\,\chi-1\right)\tau^3 + O(\tau^5) \tag{29}$$

where $\tau = \tanh H$. Baker[25] has shown that the Yang-Lee theorem implies that

$$F(\tau) = \int_0^\infty \frac{\tau\, d\psi(\omega)}{1+\tau^2\omega} = f_1\tau - f_2\tau^3 + \cdots,\ d\psi \geq 0, \tag{30}$$

i.e., $F(\tau)$ is τ times a series of Stieltjes. By standard theory[26], we must have

$$F(\tau) \geq \frac{f_1\tau}{1+f_3\tau^2/f_1}\ ,\ 0 \leq \tau \leq \infty. \tag{31}$$

If we choose,

$$\tau_s^{\,2} = f_1/f_3\ , \tag{32}$$

since $\frac{\partial^2\chi}{\partial H^2}$ is negative and dominates[27] χ, then we have, as $\tau_s \to 0$ for $\delta > 1$, by monotonicity of the magnetization in temperature,[6]

$$M(\tau_s) \sim B\tau_s^{\,1/\delta} \geq F(\tau_s) \geq \frac{1}{2}\,\chi^{3/2}\Big/\left(\frac{-\partial^2\chi}{\partial H^2}\right)^{1/2}, \tag{33}$$

which becomes,

$$\mu(T-T_c)^{\Delta/\delta} \geq \left[B_+^{3/2}/(2A_+^{\frac{1}{2}})\right](T-T_c)^{\Delta-\gamma} \tag{34}$$

or

$$\delta \geq \Delta/(\Delta-\gamma) \ . \tag{35}$$

This result is slightly stronger than the corresponding result of Gaunt and Baker[5] because their result is for Δ_∞, and this one is for Δ_4. The subscripts refer to the order of the derivative with respect to H involved in the definition. The result with Δ_4 is stronger than that with Δ_∞ as[25] $\Delta_{2m+2} \geq \Delta_{2m}$.

We have reduced the theoretical study of the apparent failure of hyperscaling in d = 3, 4 dimensions to a study of (25) which is defined in terms of only one and two-point correlations rather than (28) which also involves 4-point correlations. Presumably one could as well study the single-temperature relation, which involves only one and two-point correlations

$$2 - \eta \leq d\ (\delta-1)/(\delta+1) \tag{36}$$

which is equivalent to (25) if eq. (7) holds, but we have not proven this further simplification.

The failure of critical index, relations between correlation functions involving a different number of points is expected[28] to introduce, minimally, an anomalous dimension of the vacuum, i.e., replace d by $d-\omega^*$ in (8), and suggests that the genesis of the breakdown of hyperscaling comes in local properties at spin separations $r \ll \xi$, rather than sums over the whole lattice.

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